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A linearisation method for non-linear singular boundary value problems

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ABSTRACT

This paper presents a novel numerical approach for the solution of a class of nonlinear singular boundary value problems arising in physiology. The approach is based on a new application of the successive linearisation method (SLM). Three illustrative examples are presented to demonstrate the effectiveness of the proposed method. The new approach is found to give accurate results comparable to results in the literature found using existing numerical methods.

1. Introduction

The aim of this paper is to introduce a novel approach for the numerical solution of a class of singular two-point boundary value problems that arise in the study of tumour growth in physiology, Adam [1,2].

Consider the following class of non-linear singular boundary value problem based on reaction–diffusion equations;

$$y''(x) + \left(a + \frac{b}{x}\right) y'(x) = f(x, y), \quad 0 \le x \le 1, \tag{1}$$

$$\eta_1 y(0) + \zeta_1 y'(0) = \gamma_1, \tag{2}$$

$$\eta_2 y(1) + \zeta_2 y'(1) = \gamma_2, \tag{3}$$

where $\eta_1, \eta_2 > 0$, $\zeta_2 \ge 0$ and γ_1 and γ_2 are finite constants. The constraints often imposed on $f(x, y)$ (see [3]) are that $f(x, y) \in [0, 1] \times R$ is continuous, $\partial f/\partial y$ exists, is continuous and nonnegative for all $(x, y) \in [0, 1] \times R$. The results of this mathematical model are used to make several biological inferences.

The majority of recent studies have solved such boundary value problems using approaches based on cubic splines methods. Abukhaled et al. [3] used cubic *B*-splines to find approximate solutions of the singular two point boundary value problem above. To circumvent the singularity at the origin, they used two approaches: L'Hospital's rule and an economised Chebyshev polynomial in the vicinity of the singular point. Compared to the finite difference technique used by Pandey and Singh [4], they showed that this approach gave second order accuracy. Çağlar et al. [5] used third-degree *B*-spline functions and the Levenberg–Marquardt optimisation method to compute approximations to the solution of Eq. (1). Solutions were obtained for three physical model problems: (i) thermal explosions, (ii) oxygen diffusion, and (iii) the non-linear heat conduction model of the human head. They showed that the method either converged to the exact solution or gave small absolute errors. Related studies that used cubic spline methods include Kanth and Bhattacharya [6], Khuri and Sayfy [7] and Rashidinia et al. [8] who used a non-polynomial cubic spline method to solve a class of non-linear singular ordinary differential equations arising in physiology. A review and survey of the literature on spline and *B*-spline methods for finding solutions of singular boundary value problems is given by Kumar and Gupta [9].

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Barrea and Turner [10] studied radially symmetric cases of the general model and used a spectral numerical method to solve the system of equations. In [11], the evolution of a single tumour was discussed and the Crank–Nicolson method was used to solve the reaction–diffusion equations. Other significant contributions to the mathematical literature with regard to the solution of the tumour growth two-point boundary value problem include Chawla et al. [12] who used a finite difference method for this class of problems. Finite differences were also used by, among others, Jain and Jain [13] and Pandey and Singh [4].

In addition, analytical methods, particularly the Adomian Decomposition Method (ADM) have been used to solve nonlinear singular BVP of type (1)–(3), (see [14–17]).

In this work, we employ a new implementation of the successive linearisation method (SLM) for solving the singular BVPs given by Eqs. (1)–(3). The SLM is a very effective linearisation algorithm that was recently introduced by Motsa [18] for solving non-linear BVPs related to heat transfer. The method has also been successfully used to solve fluid mechanics related problems defined by boundary layer similarity equations in [19–21]. In this work, we extend the application of the SLM to singular BVPs of type (1)–(3).

2. Solution method

In this section, the implementation of the successive linearisation method (SLM) in solving the governing singular boundary value problem (1)–(3) is discussed. The SLM (see for example [18] for details) is based on transforming the governing nonlinear boundary value problem into an iterative scheme made up of linear differential equations which are subsequently solved using analytical or numerical methods wherever possible. To solve Eqs. (1)–(3) using the SLM it is convenient to introduce the boundary condition at $x = 0$ as

$$y(0) = \alpha \tag{4}$$

where α is a constant to be determined. The governing Eq. (1) is solved subject to the boundary conditions (2) and (4) and the boundary condition (3) is used as an extra condition required to solve for the unknown constant α. We seek a solution of the form

$$y(x) = y_i(x) + \sum_{m=0}^{i-1} y_m(x), \qquad \alpha = \alpha_i + \sum_{m=0}^{i-1} \alpha_m, \quad i = 1, 2, 3, \ldots, \tag{5}$$

where α_i and y_i are obtained iteratively by solving the linearised equations that result from substituting (5) in (1)–(3) using $y_0(x)$ and α_0 as initial approximations. Substituting (5) in (1)–(3) and neglecting nonlinear terms in y_i and α_i gives

$$y_i'' + a_{1,i-1} y_i' + a_{2,i-1} y_i = r_{i-1}, \tag{6}$$

subject to

$$y_i(0) - \alpha_i = s_{i-1}, \tag{7}$$

$$\eta_1 y_i(0) + \zeta_1 y_i'(0) = p_{i-1}, \tag{8}$$

$$\eta_2 y_i(1) + \zeta_2 y_i'(1) = q_{i-1}, \tag{9}$$

where

$$a_{1,i-1} = a + \frac{b}{x}, \qquad a_{2,i-1} = -\frac{\partial f}{\partial y}\left(x, \sum_{m=1}^{i-1} y_m\right), \tag{10}$$

$$r_{i-1} = f\left(x, \sum_{m=1}^{i-1} y_m\right) - \sum_{m=1}^{i-1} y_m'' - \left(a + \frac{b}{x}\right) \sum_{m=1}^{i-1} y_m', \tag{11}$$

$$s_{i-1} = \sum_{m=1}^{i-1} \alpha_m - \sum_{m=1}^{i-1} y_m(0), \tag{12}$$

$$p_{i-1} = \gamma_1 - \eta_1 \sum_{m=1}^{i-1} y_m(0) - \zeta_1 \sum_{m=1}^{i-1} y_m'(0), \tag{13}$$

$$q_{i-1} = \gamma_2 - \eta_2 \sum_{m=1}^{i-1} y_m(1) - \zeta_2 \sum_{m=1}^{i-1} y_m'(1). \tag{14}$$

A good initial approximation for α and $y(x)$ (given by α_0 and $y_0(x)$) can be obtained by looking for a solution of the form

$$y_0 = \alpha_0 + c_1 x + c_2 x^2 \tag{15}$$

where c_1, c_2 are constants that are obtained by evaluation of (15) using boundary conditions (2) and (3) which are given by

$$c_1 = \frac{-\alpha_0\eta_1 + \gamma_1}{\zeta_1}, \qquad c_2 = \frac{\eta_1\eta_2\alpha_0 - \eta_2\alpha_0\zeta_1 - \eta_2\gamma_1 + \zeta_2\eta_1\alpha_0 - \zeta_2\gamma_1 + \gamma_2\zeta_1}{\zeta_1(\eta_2 + 2\zeta_2)}. \tag{16}$$

The approximate value of α_0 is obtained by substituting (15) in the governing differential equation (1) and evaluating the resulting equation at a value of x (say $x = 1/2$) inside the domain of the problem. If the resulting equation cannot be solved exactly then nonlinear equation solvers that are available in scientific computing software such as `fsolve` in Maple or Matlab can be used.

To solve the linearised system (6)–(9), we use the Chebyshev collocation spectral method in which the solution space is discretised using the Chebyshev–Gauss–Lobatto collocation points

$$z_j = \cos\left(\frac{\pi j}{N}\right), \quad j = 0, 1, \dots, N \tag{17}$$

which are the extrema of the Nth order Chebyshev polynomial

$$T_N(z) = \cos(N\cos^{-1} z). \tag{18}$$

Before applying the spectral method, it is convenient to transform the governing physical region [0, 1] for the problem to the interval [−1, 1] on which the spectral method is defined. This can be achieved by using the linear transformation $x = (z + 1)/2$. The Chebyshev spectral collocation method (see for example [22–25]) is based on the idea of introducing a differentiation matrix D which is used to approximate the derivatives of the unknown variables y_i at the collocation points as the matrix vector product

$$\frac{dy_i}{dz} = \sum_{k=0}^{N} D_{jk} y_i(z_k) = D\mathbf{y}_i, \quad j = 0, 1, \dots, N \tag{19}$$

where $\mathbf{y}_i$ is the vector function at the collocation points z_j and D is the derivative matrix. The entries of D can be computed in different ways (see for example [22–25]). In this work, we use the method proposed by Trefethen [24] in the `cheb.m` Matlab m-file. Thus, applying the spectral method, with derivative matrices on the linear boundary value system (6)–(9) leads to the following linear matrix system

$$\mathbf{A}_{i-1}\mathbf{y}_i = \mathbf{r}_{i-1} \tag{20}$$

with the boundary conditions

$$y_i(z_N) - \alpha_i = s_{i-1}(z_N), \tag{21}$$

$$\eta_1 y_i(z_N) + \zeta_1 \sum_{k=0}^{N} D_{Nk} y_i(z_k) = p_{i-1}(z_N), \tag{22}$$

$$\eta_2 y_i(z_0) + \zeta_2 \sum_{k=0}^{N} D_{0k} y_i(z_k) = q_{i-1}(z_0), \tag{23}$$

where

$$\mathbf{A}_{i-1} = D^2 + \mathbf{a}_{1,i-1} D + \mathbf{a}_{2,i-1} \tag{24}$$

and $\mathbf{a}_{1,i-1}, \mathbf{a}_{2,i-1}$ are $(N + 1) \times (N + 1)$ diagonal matrices with $a_{1,i-1}(z_j), a_{2,i-1}(z_j)$ on the main diagonal and

$$\mathbf{y}_i = y_i(z_j), \qquad \mathbf{r}_i = r_i(z_j), \quad j = 0, 1, \dots, N. \tag{25}$$

The equation system (20)–(23) can be written as the following matrix equation

$$\left[\begin{array}{ccccc|c} \eta_2 + \zeta_2 D_{00} & \zeta_2 D_{01} & \cdots & \zeta_2 D_{0N-1} & \zeta_2 D_{0N} & 0 \\ & & & & & 0 \\ & & \mathbf{A}_{i-1} & & & \vdots \\ & & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ \hline \zeta_1 D_{N0} & \zeta_1 D_{N1} & \cdots & \zeta_1 D_{NN-1} & \eta_1 + \zeta_1 D_{NN} & 0 \end{array}\right] \left[\begin{array}{c} y_i(z_0) \\ y_i(z_1) \\ \vdots \\ y_i(z_{N-1}) \\ y_i(z_N) \\ \hline \alpha_i \end{array}\right] = \left[\begin{array}{c} q_{i-1}(z_0) \\ r_{i-1}(z_1) \\ \vdots \\ r_{i-1}(z_{N-1}) \\ s_{i-1} \\ \hline p_{i-1}(z_N) \end{array}\right]. \tag{26}$$

The boundary conditions (21) and (23) have been imposed on the first and last rows of $\mathbf{A}_{i-1}$ and $\mathbf{r}_{i-1}$ whilst the boundary condition (22) has been added as the $(N+2)$ row. Thus, starting from the initial approximations $y_0(x)$ and α_0, the subsequent solutions for y_i and α_i $(i = 1, 2, 3 \dots)$ can be obtained by solving the matrix system (26).

3. Numerical experiments

In this section, we use the general SLM solution procedure outlined in the previous section to solve specific problems that are governed by Eqs. (1)–(3). In order to assess the performance of the method, we consider three sample boundary value problems that were previously solved using other numerical and analytical methods.

Example 3.1. Consider the following singular boundary value problem of the form

$$y''(x) + \frac{2}{x}y'(x) = -\lambda e^{-ky}, \tag{27}$$

$$y'(0) = 0, \qquad B_1 y(1) + y'(1) = B_2, \tag{28}$$

which arises in the modelling of the distribution of heat sources in the human head [5,7,16,8].

In this problem, the parameters used in the SLM algorithm are

$$f(x, y) = -\lambda e^{-ky}, \qquad a_{1,i-1} = \frac{2}{x}, \qquad a_{2,i-1} = -\lambda k \exp\left[\sum_{m=0}^{i-1} y_m\right], \tag{29}$$

$$r_{i-1} = -\lambda \exp\left[\sum_{m=0}^{i-1} y_m\right] - \sum_{m=0}^{i-1} y''_m - \frac{2}{x}\sum_{m=0}^{i-1} y'_m. \tag{30}$$

Example 3.2. Consider the following singular boundary value problem [5,7,8] of the form

$$y''(x) + \frac{2}{x}y'(x) = \frac{ny}{y+k}, \tag{31}$$

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5, \tag{32}$$

which arises in the modelling of oxygen diffusion in spherical cells with Michaelis–Menten uptake kinetics. To facilitate comparison with numerical results in the literature (see, for example, [5,7,8]), we take $n = 0.76129$ and $k = 0.03119$ for our numerical simulations.

In this problem, the parameters used in the SLM algorithm are

$$f(x, y) = \frac{ny}{y+k}, \qquad a_{1,i-1} = \frac{2}{x}, \qquad a_{2,i-1} = -\frac{nk}{\left(\sum\limits_{m=0}^{i-1} y_m + k\right)^2}, \tag{33}$$

$$r_{i-1} = \frac{n\sum\limits_{m=0}^{i-1} y_m}{\sum\limits_{m=0}^{i-1} y_m + k} - \sum_{m=0}^{i-1} y''_m - \frac{2}{x}\sum_{m=0}^{i-1} y'_m. \tag{34}$$

Example 3.3. Consider the following singular boundary value problem [17,4,8]

$$y''(x) + \left(1 + \frac{b}{x}\right)y'(x) = \frac{5x^3(5x^5e^y - x - b - 4)}{4+x^5}, \tag{35}$$

$$y'(0) = 0, \qquad y(1) + 5y'(1) = \ln\left(\frac{1}{5}\right) - 5, \tag{36}$$

with exact solution $y(x) = \ln(\frac{1}{4+x^5})$.

In this problem, the parameters used in the SLM algorithm are

$$f(x, y) = \frac{5x^3(5x^5e^y - x - m_1 - 4)}{4+x^5}, \qquad a_{1,i-1} = 1 + \frac{m_1}{x}, \qquad a_{2,i-1} = -\frac{25x^8 \exp\left[\sum\limits_{m=0}^{i-1} y_m\right]}{4+x^5}, \tag{37}$$

$$r_{i-1} = \frac{5x^3\left(5x^5 \exp\left[\sum\limits_{m=0}^{i-1} y_m\right] - x - m_1 - 4\right)}{4+x^5} - \sum_{m=0}^{i-1} y''_m - \left(1 + \frac{m_1}{x}\right)\sum_{m=0}^{i-1} y'_m. \tag{38}$$

Table 1
Example 3.1: comparison of the ADM results of Makinde [16] and the present SLM results when $\lambda = 1, B_1 = B_2 = 1, k = 1$.

ADM results [16]		Present results	
Iter.	$y(0)$	Iter.	$y(0)$
2	1.1607192415883	1	1.1605789506935
6	1.1608198195214	2	1.1608198169433
8	1.1608198195901	3	1.1608198195901
10	1.1608198195901	4	1.1608198195901

Table 2
Example 3.1: comparison of the present SLM results for $y(0)$ against results from the literature when $\lambda = 1, B_1 = 0.1, B_2 = 0, k = 1, N = 30$.

Iter.	Present results	Other methods	
1	1.1333364796133	1.14703993670271	Third-degree B-spline[5]
2	1.1469908947186	1.14704108351547	Ordinary cubic spline method [6]
3	1.1470390187374	1.14704079519111	Cubic B-spline collocation [7]
4	1.1470390193298	0.1147039160	Finite differences [4]
5	1.1470390193298	0.11470486854	Non-polynomial cubic splines [8]

Table 3
Example 3.1: comparison of the present SLM results for $y(0)$ against results from the literature when $\lambda = 1, B_1 = 1, B_2 = 0, k = 1, N = 30$.

Iter.	Present results	Other methods	
1	0.8284448286382	0.82848327295802	Third-degree B-spline[5]
2	0.8284832903548	0.82848327300049	Ordinary cubic spline method [6]
3	0.8284832903597	0.82848329481355	Cubic B-spline collocation [7]
4	0.8284832903597	0.8284831497	Finite differences [4]
5	0.8284832903597	0.828483273	Non-polynomial cubic splines [8]

Table 4
Comparison of the present SLM results for Example 3.2 against the numerical results of [5] and [8].

x	Present results	Ref. [5]	Ref. [8]
0	0.82848329035969	0.82848327295802	0.8284833089
0.1	0.82970609243327	0.82970607521884	0.8297060779
0.2	0.83337473358953	0.83337471691089	0.8333747471
0.3	0.83948991395280	0.83948989814383	0.8394899001
0.4	0.84805278499483	0.84805277036165	0.8480527729
0.5	0.85906492716923	0.85906491397434	0.8590649189
0.6	0.87252831995724	0.87252830841853	0.8725283096
0.7	0.88844530562258	0.88844529589927	0.8884452969
0.8	0.90681854806607	0.90681854026297	0.9068185417
0.9	0.92765098836536	0.92765098252660	0.9276509838
1	0.95094579849648	0.95094579461056	0.9509457969

4. Results

In this section, we present the approximate solutions of the illustrative examples using the successive linearisation method (SLM). In order to assess the performance and reliability of the present method of solution the results are presented in Tables 1–6 and compared with results in the literature obtained using other numerical methods.

In Table 1, we present the convergence of the SLM results for $y(0) = \alpha_i$. The findings are compared with Makinde [16] who used the Adomian decomposition method (ADM) to solve the same problem. Evidently only three iterations are required to obtain results that are accurate to at least thirteen decimal places. In the ADM approach of Makinde [16], eight iterations were required to get the same level of accuracy.

In Table 2, we give an indication of how the SLM results converge to the solution of $y(0)$ at each iteration when $\lambda = 1$, $B_1 = 0.1$, $B_2 = 0$ and $k = 1$. The results are compared with results generated using other numerical methods in the literature. We note that the SLM results converge to a fixed value after only four iterations. The results reported in [5–7] match the SLM results to only five decimal places. We note also that the results presented in [4,8] should have 1 in place of zero as first digit.

In Table 3, we give a comparison of the SLM results for $y(0)$ against the results of [5–7,4,8] when $\lambda = 1, B_1 = 1, B_2 = 0$, $k = 1$. The SLM results converge to a fixed value after only three iterations.

Table 5
Example 3.3: comparison of the present SLM results for y(0) against the exact value of −1.3862943611199 for different values of b.

Iter. \ b	$b = 0.25$	$b = 0.75$	$b = 1$	$b = 2$
1	−0.9400145702880	−0.9365998542030	−0.9400969900064	−0.9650634161365
2	−1.3459218856813	−1.3464421717189	−1.3476444986948	−1.3539079170985
3	−1.3859964648192	−1.3860138987313	−1.3860350346998	−1.3861239877666
4	−1.3862943450903	−1.3862943473962	−1.3862943495839	−1.3862943564552
5	−1.3862943611199	−1.3862943611199	−1.3862943611199	−1.3862943611199
6	−1.3862943611199	−1.3862943611199	−1.3862943611199	−1.3862943611199

Table 6
Maximum absolute errors for Example 3.3 for $b = 0.25, 0.75$.

b	Iter.	ADM [17]	Iter.	SLM ($N = 30$)	Finite diff. [4]	Cubic spline [8]
0.25	5	6.83×10^{-6}	5	2.67×10^{-15}	1.20×10^{-6}	3.00×10^{-8}
0.75	5	5.75×10^{-6}	5	1.75×0^{-14}	1.36×10^{-6}	1.17×10^{-6}

Table 7
Example 3.3: comparison of the present SLM results for the maximum absolute errors for $b = 1$, $b = 2$ against the cubic spline results of Rashidiana et al. [8].

N	Present results		Rashidiana et al. [8]	
	$b = 1$	$b = 2$	$b = 1$	$b = 2$
16	1.55(−08)	1.70(−08)	4.66(−05)	7.46(−05)
32	6.88(−15)	1.53(−14)	4.95(−06)	7.06(−06)
64	7.33(−15)	8.66(−15)	4.92(−07)	6.85(−06)
128	7.75(−14)	9.75(−14)	4.87(−07)	6.75(−07)
256	4.49(−13)	4.12(−13)	4.84(−08)	6.71(−08)
512	9.62(−13)	9.52(−13)	4.82(−08)	6.69(−08)

Table 4 gives a comparison between the SLM results for $y(x)$ obtained after five iterations against the B-spline numerical results of [5] and the non-polynomial cubic spline results of [8]. It can be seen from Table 5 that the present results match those of [5,8] up to seven decimal places. We note that the maximum absolute errors for the methods used in [5,8] are analysed for other singular BVPs with exact solutions (such as Example 3.3) and it was found that even for very small step sizes, the errors were never smaller than 1×10^{-8}.

In Table 5, we show the convergence of the SLM approach towards the exact solution of Example 3.3 at $x = 0$ using different values of b. We observe that full convergence to thirteen decimal place accuracy is achieved after only five iterations for all the values of b considered.

Table 6 gives a comparison between the SLM maximum absolute errors for Example 3.3 against results from other solution methods. Example 3.3 was solved using the finite difference by Pandey and Singh [26]. Later, the same authors solved the problem using an improved version of their finite difference method, [4]. The problem was also solved by Mittal and Nigam [17] using the Adomian decomposition method. More recently, Rashidinia et al. [8] introduced a slightly more accurate approach based on non-polynomial cubic splines to solve the same problem. The finite difference and cubic spline results given in Table 6 correspond to the maximum error generated using $N = 512$ grid points in [4,8]. It can be seen that the present approach is more effective than the methods used in [4,8]. It leads to more accurate results using far fewer grid points ($N = 30$) than the finite difference and cubic spline methods. When the same number of iterations are considered, the present method appears to give better accuracy compared to the ADM approach of Mittal and Nigam [17].

In Table 7, we give a comparison of the SLM results after five iterations for the maximum absolute errors for $b = 1$, $b = 2$ against the cubic spline results of Rashidiana et al. [8]. The present approach gives more accurate results than the cubic spline results when the same number of grid points are used.

5. Conclusion

In this paper, we presented a novel application of the successive linearisation method (SLM) to a class of singular boundary value problems that arise in physiology. We proposed a general numerical scheme based on the SLM that can be used to solve many problems that belong to the class of singular BVPs discussed in this work. A comparison was made between exact analytical solutions, numerical results from the literature and the present approximate solutions. The numerical results indicate that our method converges rapidly to the exact solutions. Nonetheless, at this stage we have only been able to assess the rate of convergence of the method through numerical experimentation and simulations using selected examples. A more rigorous mathematical theory behind the method still remains to be developed.

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